Objectives:

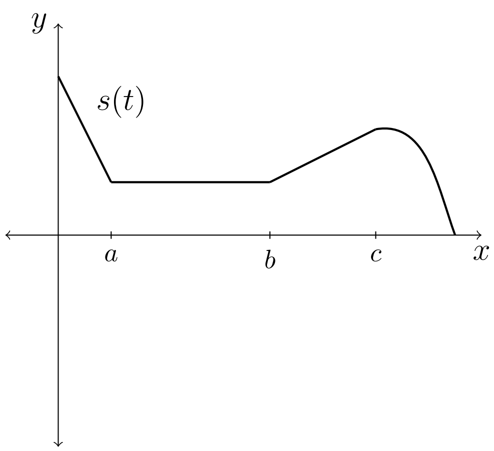
- Determine whether or not a given function is differentiable
- Compute derivatives of functions
- Recognize when an expression is the derivative of another function

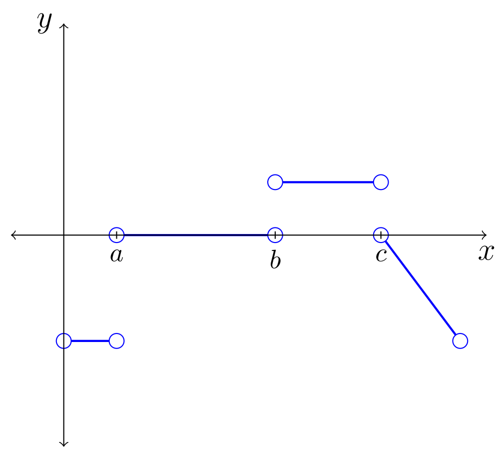
In the last project, we worked with the derivative function:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}, \quad \text{alternatively,} \quad f'(x) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

Example: Given the graph of the position of an object, $s(t)$, graph the velocity of the object, $v(t)$.

Note: $v(t) =$ s'(t).





Example: This limit is the derivative of which function?

$$\lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} = f'(x) \text{ where } f(x) = \sin(x)$$

Example: What about this limit? Is it a derivative for some function?

$$\lim_{h\to 0} \frac{e^h - 1}{h} = f'(0) \text{ where } f(x) = e^x$$

Notation: Given a function $y = f(x)$, we write the derivative as $f'(x)$ or $\frac{dy}{dx}$.

Higher order derivatives: We can find the derivative of $f'(x)$ which is called a "second derivative" and is written $f''(x)$ or $\frac{d^2y}{dx^2}$.

If $s(t)$ is a function representing position over time, then $s'(t)$ gives velocity and $s''(t)$ gives acceleration. Also, $s'''(t)$ is called jerk.

Non-differentiable functions:

- If $f(x)$ is not continuous, then it is not differentiable.
- If $f(x)$ has a vertical tangent line at any point, it is not differentiable (ex. $x^{1/3}$).
- If $f(x)$ has a corner, it is not differentiable (ex. $|x|$).

Example: Prove that $g(x) = x^{1/3}$ is not differentiable at $x = 0$.

We need to compute $\lim_{h \to 0} \frac{g(0+h) - g(0)}{h}$.

$$\lim_{h \to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0} \frac{h^{1/3} - 0}{h} = \lim_{h \to 0} \frac{h^{1/3}}{h} = \lim_{h \to 0} \frac{1}{h^{2/3}} = \infty$$

Since $\lim_{h \to 0} \frac{g(0+h) - g(0)}{h}$ D.N.E., we know $g(x) = x^{1/3}$ is not differentiable at $x = 0$.

Fact: A function f that is differentiable at a is also continuous at a.

Proof: Suppose that $f'(a)$ exists. Using the alternate definition of derivative, $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists. We must show that $f(x)$ is continuous at $x = a$ which means $\lim_{x \to a} f(x) = f(a)$. We can compute

$$\lim_{x \to a} f(x) - f(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}(x - a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a}(x - a) = f'(a) \cdot 0 = 0.$$

We have shown $\lim_{x \to a} f(x) - f(a) = 0$. So $\lim_{x \to a} f(x) = f(a)$.